JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **30**, No. 1, February 2017
http://dx.doi.org/10.14403/jcms.2017.30.1.159

A NOTE ON EXPLICIT SOLUTIONS OF CERTAIN IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS

Namjip Koo*

Abstract. This paper deals with linear impulsive differential equations involving the Caputo fractional derivative. We provide exact solutions of nonhomogeneous linear impulsive fractional differential equations with constant coefficients by means of the Mittag-Leffler functions.

1. Introduction and preliminaries

Denton and Vatsala [5] established the explicit representation of the solution of the linear fractional differential equation with variable coefficients and they developed the Gronwall integral inequality for the Riemann-Liouville fractional differential equations. Choi et al. [1] obtained an exact solution of linear Caputo fractional differential equations by the help of the Mittag-Leffler functions. Also, Choi et al. [2] studied the stability for Caputo fractional differential equations. Fečkan et al. [6] studied a Cauchy problem for a fractional differential equation with linear impulsive conditions and made a counterexample to illustrate that the concepts of piecewise continuous solutions used in current papers are not appropriate. Wang et al. [10] obtained many new existence, uniqueness and data dependence results of solutions for nonlinear impulsive fractional differential equations involving the Caputo fractional derivative via some generalized singular Gronwall inequalities. Choi and Koo

Received January 18, 2017; Accepted January 27, 2017.
2010 Mathematics Subject Classification: Primary 26A33, 34A08, 34A37.
Key words and phrases: impulsive fractional differential equation, Caputo fractional derivative, Mittag-Leffler function.
Correspondence should be addressed to Namjip Koo, njkoo@cnu.ac.kr.
This work was supported by research fund of Chungnam National University in 2016.

[3, 4] obtained exact solutions for linear impulsive fractional differential equations with constant coefficients by means of the Mittag-Leffler functions.

In this paper we provide exact solutions of nonhomogeneous linear impulsive fractional differential equations with constant coefficients by means of the Mittag-Leffler functions.

We recall the notion of Mittag-Leffler functions which were originally introduced by G. M. Mittag-Leffler(see [9]). That is, one parameter family Mittag-Leffler function is given by

$$E_q(t^q) = \sum_{k=0}^{\infty} \frac{t^{qk}}{\Gamma(qk+1)},\ q > 0$$

and two parameter family Mittag-Leffler function is defined as

$$E_{q,r}(t^q) = \sum_{k=0}^{\infty} \frac{t^{qk}}{\Gamma(qk+r)},\ q, r > 0,$$

where Γ is the Gamma function given by

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt,\ Re(z) > 0.$$

The Mittag-Leffler functions which are the generalizations of the exponential function play an important role in the theory of fractional differential equations.

Let q be a positive real number such that $0 < q \leq 1$ and $t_0, T \in [0, \infty)$. We recall the definition of the Caputo fractional derivative of a function $u : [t_0, \infty) \to \mathbb{R}$.

Definition 1.1. [7] The *Caputo fractional derivative of order* q of a function u is defined by

$$^{C}D_{t_0}^{q} u(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} u'(s) ds,$$

where $u'(t) = \frac{du(t)}{dt}$.

For the fractional calculus and the theory of fractional differential equations, we refer the reader to [7].

Throughout this paper, let $J = [t_0, T]$. Assume that $\{t_k\}_{k=1}^{m}$ satisfies $0 \leq t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $u(t_k^+) = \lim_{\varepsilon \to 0^+} u(t_k + \varepsilon)$ and $u(t_k^-) = \lim_{\varepsilon \to 0^-} u(t_k + \varepsilon)$ represent the right and left limits of $u(t)$ at $t = t_k$. Denote by $C(J, \mathbb{R})$ the set of all continuous functions from J

into $\mathbb{R}$. Also, let $PC(J,\mathbb{R})$ be the set of all functions from J into $\mathbb{R}$ as follows:

$$PC(J,\mathbb{R}) = \{u : J \to \mathbb{R} | u \in C((t_k, t_{k+1}], \mathbb{R}), k = 0, 1, \cdots, m, \text{ and there exist } u(t_k^-) \text{ and } u(t_k^+), k = 1, \cdots, m, \text{ with } u(t_k^-) = u(t_k)\}.$$

We consider the following fractional Cauchy problem

$$\begin{cases} {}^C D_{t_0}^q u(t) = f(t, u(t)), t \neq t_k, t \in J, \\ \Delta u(t_k) = I_k(u(t_k^-)),\ k = 1, 2, \cdots, m, \\ u(t_0) = u_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$ is jointly continuous, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, and $I_k : \mathbb{R} \to \mathbb{R}$. For the concept and existence of solution for Eq. (1.1), see [6, 10]. Also, for the general theory and applications of impulsive differential equations, we refer the reader to [8].

2. Main results

In this section we give exact solutions of nonhomogeneous linear impulsive fractional differential equations with constant coefficients by the means of the Mittag-Leffler functions.

Lemma 2.1. [10] *Let a be a real number with $a > t_0$. Then a function $u \in C(J,\mathbb{R})$ is a solution of the following fractional Cauchy problem*

$$\begin{cases} {}^C D_{t_0}^q u(t) = f(t, u(t)), t \in J, \\ u(a) = u_0 \end{cases} \tag{2.1}$$

if and only if it is a solution of the following fractional integral equation

$$\begin{aligned} u(t) &= u_0 - \frac{1}{\Gamma(q)} \int_{t_0}^{a} (a-s)^{q-1} f(s, u(s)) ds \\ &+ \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, u(s)) ds. \end{aligned}$$

Lemma 2.2. [10] *A function $u \in PC(J,\mathbb{R})$ is a solution of the following fractional integral equation*

$$u(t) = \begin{cases} u(t_0) + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s, u(s)) ds, t \in [t_0, t_1], \\ \vdots \\ u(t_0) + \sum\limits_{t_0 < t_k < t} I_k(u(t_k^-)) \\ + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s, u(s)) ds, t \in (t_k, t_{k+1}], k = 1, \cdots, m, \end{cases}$$

if and only if it is a solution of Eq. (1.1).

Next, we consider the nonhomogeneous linear differential equation involving the Caputo fractional derivative

$$ {}^{C}D_{t_0}^{q}x(t) = \lambda x(t) + h(t),\ x(t_0) = x_0, \tag{2.2}$$

where $x, h \in C(J, \mathbb{R})$ are continuous. Then the unique solution $x(t)$ of Eq. (2.2) satisfies the following integral equation

$$x(t) = x(t_0)E_q(\lambda(t-t_0)^q) + \int_{t_0}^{t}(t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)h(s)ds \tag{2.3}$$

for $t \geq t_0$.

LEMMA 2.3. [1, Lemma 3.2] *If we set $h(t) \equiv d$ in Eq. (2.2) with a constant d, then the solution $x(t)$ of Eq. (2.3) reduces to*

$$x(t) = x(t_0)E_q(\lambda(t-t_0)^q) + d(t-t_0)^q E_{q,q+1}(\lambda(t-t_0)^q), t \in J.$$

REMARK 2.4. If we set $h(t) \equiv d$ in Eq. (2.2) involving the Caputo fractional derivative of the order $q = 1$, then the solution $x(t)$ of of Eq. (2.3) reduces to

$$\begin{aligned} x(t) &= x(t_0)E_1(\lambda(t-t_0)) + d(t-t_0)E_{1,2}(\lambda(t-t_0)) \\ &= e^{\lambda(t-t_0)}x(t_0) + \frac{d}{\lambda}(e^{\lambda(t-t_0)} - 1),\ t \in J, \end{aligned}$$

where λ is nonzero constant.

We can obtain the following result about exact solutions of nonhomogeneous linear impulsive fractional differential equations with constant coefficients by the help of the Mittag-Leffler functions. This result is an improvement of Theorem 2.4 in [3].

THEOREM 2.5. *If we set $f(t,u) \equiv \lambda u + d$ with constants λ, d and $I_k(u(t_k^-)) = \beta_k u(t_k^-), k = 1, 2, \cdots, m$, with each constant β_k in Eq. (1.1), then the solution $u(t)$ of Eq. (1.1) is given by*

$$u(t) = \begin{cases} u_0E_q(\lambda(t-t_0)^q) + d(t-t_0)^qE_{q,q+1}(\lambda(t-t_0)^q)),\ t \in [t_0, t_1], \\ u_0E_q(\lambda(t-t_k)^q)\prod_{i=1}^{k}(1+\beta_i)E_q(\lambda(t_i - t_{i-1})^q) \\ +dE_q((\lambda(t-t_k)^q))\sum_{j=1}^{k}(t_j - t_{j-1})^qE_{q,q+1}(\lambda(t_j-t_{j-1})^q)\times \\ \prod_{i=j}^{k}((1+\beta_i))\prod_{i=j+1}^{k}E_q(\lambda(t_i-t_{i-1})^q) \\ +d(t-t_k)^qE_{q,q+1}(\lambda(t-t_k)^q),\ t \in (t_k, t_{k+1}], \end{cases}$$

where $k = 1, 2, \cdots, m$ and $\prod_{i=k+1}^{k}E_q(\lambda(t_k + 1 - t_k)^q) = 1$.

Proof. Let $t \in [t_0, t_1]$. Then it follows from Lemma 2.3 that

$$u(t) = u_0 E_q(\lambda(t-t_0)^q) + d(t-t_0)^q E_{q,q+1}(\lambda(t-t_0)^q),\ t \in [t_0, t_1].$$

If $t \in (t_1, t_2]$, then we obtain

$$\begin{aligned} u(t) &= (1+\beta_1)u(t_1^-)E_q(\lambda(t-t_1)^q) + d(t-t_1)^q E_{q,q+1}(\lambda(t-t_1)^q) \\ &= u_0(1+\beta_1)E_q(\lambda(t_1-t_0)^q)E_q(\lambda(t-t_1)^q) \\ &+ (1+\beta_1)d(t_1-t_0)^q E_{q,q+1}(\lambda(t_1-t_0)^q)E_q(\lambda(t-t_1)^q) \\ &+ d(t-t_1)^q E_{q,q+1}(\lambda(t-t_1)^q),\ t \in (t_1, t_2]. \end{aligned}$$

If $t \in (t_2, t_3]$, then we obtain

$$\begin{aligned} u(t) &= (1+\beta_2)u(t_2^-)E_q(\lambda(t-t_2)^q) + d(t-t_2)^q E_{q,q+1}(\lambda(t-t_2)^q) \\ &= u_0(1+\beta_1)(1+\beta_2)E_q(\lambda(t_1-t_0)^q)E_q(\lambda(t_2-t_1)^q)E_q(\lambda(t-t_2)^q) \\ &+ (1+\beta_1)(1+\beta_2)d(t_1-t_0)^q E_{q,q+1}(\lambda(t_1-t_0)^q)E_q(\lambda(t_2-t_1)^q) \times \\ &\quad E_q(\lambda(t-t_2)^q) + (1+\beta_2)d(t_2-t_1)^q E_{q,q+1}(\lambda(t_2-t_1)^q)E_q(\lambda(t-t_2)^q) \\ &+ d(t-t_2)^q E_{q,q+1}(\lambda(t-t_2)^q),\ t \in (t_2, t_3]. \end{aligned}$$

Let $t \in (t_k, t_{k+1}]$. Then it follows from above similar argument that

$$\begin{aligned} & u(t) = (1+\beta_k)u(t_k^-)E_q(\lambda(t-t_k)^q) + d(t-t_k)^q E_{q,q+1}(\lambda(t-t_k)^q) \\ & = u_0 \prod_{i=1}^{k}(1+\beta_i)E_q(\lambda(t_i-t_{i-1})^q)E_q(\lambda(t-t_k)^q) \\ & + \prod_{i=1}^{k}(1+\beta_i)d(t_1-t_0)^q E_{q,q+1}(\lambda(t_1-t_0)^q)\prod_{i=2}^{k} E_q(\lambda(t_i-t_{i-1})^q)E_q(\lambda(t-t_k)^q) \\ & + \prod_{i=2}^{k}(1+\beta_i)d(t_2-t_1)^q E_{q,q+1}(\lambda(t_2-t_1)^q)\prod_{i=3}^{k} E_q(\lambda(t_i-t_{i-1})^q)E_q(\lambda(t-t_k)^q) \\ & + \cdots + \prod_{i=k-1}^{k}(1+\beta_i)d(t_{k-1}-t_{k-2})^q E_{q,q+1}(\lambda(t_{k-1}-t_{k-2}) \times \\ & \quad \prod_{i=k}^{k} E_q(\lambda(t_i-t_{i-1})^q)E_q(\lambda(t-t_k)^q) \\ & + \prod_{i=k}^{k}(1+\beta_i)d(t_k-t_{k-1})^q E_{q,q+1}(\lambda(t_k-t_{k-1})E_q(\lambda(t-t_k)^q) \\ & + d(t-t_k)^q E_{q,q+1}(\lambda(t-t_k)^q),\ t \in (t_k, t_{k+1}], \end{aligned}$$

where $k = 1, \cdots, m$. This completes the proof. □

In view of Theorem 2.5, we can obtain the following result in [4, Theorem 3.4].

Corollary 2.6. *If* $f(t,u) = \lambda u$ *and* $I_k(u(t_k^-)) = \beta_k u(t_k^-)$ *with constants* λ *and* β_k *in Eq. (1.1) for* $k = 1, \cdots, m$*, then the solution* $u(t)$ *of Eq. (1.1) reduces to*

$$u(t) = \begin{cases} u_0 E_q(\lambda(t-t_0)^q), t \in [t_0, t_1], \\ u_0 \prod_{i=1}^{k}[(1+\beta_i)E_q(\lambda(t_i - t_{i-1})^q)]E_q(\lambda(t-t_k)^q), t \in (t_k, t_{k+1}], \end{cases} \tag{2.4}$$

where $k = 1, \cdots, m$*.*

Remark 2.7. In addition to the assumptions of Eq. (1.1), assume that $\{t_k\}_{k=1}^{\infty}$ satisfies $0 \le t_0 < t_1 < \cdots < t_k < \cdots$ and $\lim_{k\to\infty} t_k = +\infty$. Then we can extend Theorem 2.5 and Corollary 2.6 to case of Eq. (1.1) with $J = [t_0, \infty)$ and $\{t_k\}_{k=1}^{\infty}$.

References

[1] S. K. Choi, B. Kang, and N. Koo, Stability for Caputo fractional differential equations, *Proc. Jangjeon Math. Soc.* **16** (2013), 165-174.

[2] S. K. Choi, B. Kang, and N. Koo, Stability of Caputo fractional differential systems, *Abstr. Appl. Anal.* **2014** (2014), Article ID 631419, 6 pages.

[3] S. K. Choi and N. Koo, A note on linear impulsive fractional differential equations, *J. Chungcheong Math. Soc.* **28** (2015), 583-590.

[4] S. K. Choi and N. Koo, On exact solutions for impulsive differential equations with non-integer orders, *J. Chungcheong Math. Soc.* **29** (2016), 517-521.

[5] Z. Denton and A. S. Vatsala, Fractional integral inequalities and applications, *Comput. Math. Appl.* **59** (2010), 1087-1094.

[6] M. Fečkan, Y. Zhou, and J. Wang, On the concept and existence of solution for impulsive fractional differential equations, *Commmun. Nonlinear Sci. Numer. Simul.* **17** (2012), 3050-3060.

[7] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[8] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific Publishing Co. Pte. Ltd., NJ, 1989.

[9] G. M. Mittag-Leffler, Sur l'intégrable de Laplace-Abel, *C. R. Acad. Sci. Paris (Ser. II)* **136** (1903), 937-939.

[10] J. Wang, Y. Zhou, and M. Fečkan, Nonlinear impulsive problems for fractional differential equations and Ulam stability, *Comput. Math. Appl.* **64** (2012), 3389-3405.

*

Department of Mathematics
Chungnam National University
Daejeon 34134, Republic of Korea
E-mail: njkoo@cnu.ac.kr